EMISSION-LINE PROFILES FROM EXPANDING ENVELOPES

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ABSTRACT

We have developed a method for calculating the emission-line profiles from an expanding envelope. The method applies to an envelope where the emission intensity varies with latitude. Some typically illustrative cases are considered.

I. INTRODUCTION

There is little doubt that the breadth of the emission lines in Wolf-Rayet stars and the novae arises from the relative motion of the atoms in a shell of gas surrounding the core. The presence of catastrophic explosions, as in the case of novae, justifies the postulate of expanding shells of luminous gas which account for the observed profiles during and immediately after the outburst. Menzel[1] and Beals[2] considered the simplest case of a shell expanding with constant velocity. The profile from such an envelope is essentially flat-topped. Gerasimovič[3] studied the line profiles from an expanding gaseous shell surrounding a nova whose velocity of expansion is a function of the radius and whose emission at a point depends on the density. Chandrasekhar[4] extended this earlier work in an important study of expanding envelopes in general and was the first to stress the necessity of considering the "occultation effect" caused by the body of the star. Wilson[5] modified Chandrasekhar's method of analysis to facilitate the actual fitting of the observed profiles of novae to theoretical ones. We develop here a general method for the determination of an emission-line profile, based on certain specific assumptions. The analysis facilitates the easy derivation of profiles in cases characterized by lack of symmetry.

II. THE CONTOURS OF THE EMISSION BANDS

The problem has three distinct parts. The first is purely geometrical and refers to the choice of co-ordinates and assumptions regarding the atmospheric structure. The second is physical, defining the origin of luminosity and its dependence on density, velocity, etc. The third is kinematical and fixes the variation of velocity in the shell.

We adopt a conventional system of polar co-ordinates, r, θ, ϕ, with the axis of symmetry directed toward the observer. Let $i(r, \theta, \phi)$ be the emission per unit volume. Then the emission from the volume element will be

$$i(r, \theta, \phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi\, . \tag{1}$$

If this element expands radially with a uniform velocity v, the projected velocity u, in the line of sight of the observer, will be

$$u = -v \cos\theta\, . \tag{2}$$

Line-of-sight velocities toward the observer are conventionally negative. Now, since the volume element extends over $d\theta$, the velocity component will range over du, according to the law

$$du = v \sin\theta\, d\theta\, . \tag{3}$$

[1] *Pub. A.S.P.*, **41**, 344, 1929.
[2] *M.N.*, **91**, 966, 1931.
[3] *Zs. f. Ap.*, **7**, 335, 1933.
[4] *M.N.*, **94**, 522, 1934.
[5] *Ap. J.*, **80**, 259, 1934.

We can therefore change variables and say that the emission from the volume element is

$$i\left(r, \frac{u}{v}, \phi\right) r^2 dr\, d\phi \frac{du}{v}, \tag{4}$$

since, by virtue of equation (2), we can say that θ is a function of (u/v). Chandrasekhar eliminates r rather than θ, a choice that complicates slightly the selection of the integration limits. It also makes more difficult the extension of the model to nonuniform emission over the disk, a case that we shall discuss later.

If λ_0 is the position of the undisplaced line, the volume element will contribute to the emission at a new wave length, λ, such that

$$\lambda = \lambda_0 \left(1 + \frac{u}{c}\right). \tag{5}$$

Atoms in the volume element, over the range du, will contribute radiation to the wave-length interval

$$d\lambda = \lambda_0 \frac{du}{c}. \tag{6}$$

Therefore, we can say that the volume element contributes radiation at λ over the range $d\lambda$, of amount

$$i\left(r, \frac{\lambda - \lambda_0}{\lambda_0} \frac{c}{v}, \phi\right) \frac{c}{v} r^2 dr\, d\phi \frac{d\lambda}{\lambda_0}. \tag{7}$$

To get the total emission in the range $d\lambda$, we have to define i as a function of r and ϕ, allowing for possible dependence on θ by means of equations (2) and (3). We next integrate over the co-ordinates r and ϕ. Or

$$E_\lambda p\lambda = \int\int i\left(r, \frac{\lambda - \lambda_0}{\lambda_0} \frac{c}{v}, \phi\right) \frac{c}{v} r^2 dr\, d\phi \frac{d\lambda}{\lambda_0}. \tag{8}$$

Let

$$l = \frac{\lambda - \lambda_0}{\lambda_0}, \qquad dl = \frac{d\lambda}{\lambda_0}, \tag{9}$$

and measure the position within the line, so that $l = 0$ refers to the undisplaced center. Then

$$E_l dl = \int\int i\left(r, l \frac{c}{v}, \phi\right) \frac{c}{v} r^2 dr\, d\phi\, dl. \tag{10}$$

Our next concern is with the physics of the problem. We should expect that the emission in so highly ionized an atmosphere would depend on the square of the density. To achieve slightly greater generality, we shall adopt

$$i = C\rho^2 \left(\frac{v}{v_0}\right)^\beta, \tag{11}$$

where v_0 is a constant that we shall shortly define. If radial currents are responsible for the maintenance of the density distribution, so that v is radial and dependent solely upon r, we have recourse to the equation of continuity,

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{\partial \rho}{\partial t}. \tag{12}$$

For a steady state, ρ is independent of t, and we get

$$\frac{d}{dr}(\rho r^2 v) = 0 ; \tag{13}$$

and hence

$$r^2 \rho v = \text{Constant} = r_0^2 \rho_0 v_0 , \tag{14}$$

where ρ_0 and v_0 represent the values of these parameters at some specially chosen value of the radius $r = r_0$. We shall eventually identify r_0 with the effective inner boundary of a shell surrounding the star. With this equation,

$$i = C\rho_0^2 \left(\frac{r_0}{r}\right)^4 \left(\frac{v}{v_0}\right)^{\beta-2} . \tag{15}$$

As far as our equations are concerned, ρ_0 may still be a function of θ and ϕ and therefore of u and ϕ. Hence we write

$$E_l dl = \frac{C r_0^3 c}{v_0} \int\int \rho_0^2 \left(\frac{r_0}{r}\right)^2 \left(\frac{v}{v_0}\right)^{\beta-3} d\left(\frac{r}{r_0}\right) d\phi dl . \tag{16}$$

For the simplest case of all, wherein ρ_0 is independent of θ and ϕ and where $v =$ Constant $= v_0$, we get

$$E_l dl = \frac{2\pi C r_0^3 c \rho_0^2}{v_0} \left[\frac{r_0}{r}\right]_{r_{\min}}^{r_{\max}} dl . \tag{17}$$

The coefficient of dl is a constant, i.e., it is independent of l as long as

$$|l| \leqslant \frac{v_0}{c} . \tag{18}$$

If, for example, we take $r_{\min} = r_0$ and $r_{\max} \gg r_0$, for the inner and outer radii of the shell, we have

$$E_l dl = \frac{2\pi C r_0^3 \rho_0^2 c}{v_0} dl . \tag{19}$$

This analysis pertains to the previously mentioned case, first discussed by Beals, where the profile is flat-topped.

To make further progress, one must know the kinematical picture, or the dependence of v upon r. We assume the following law, which lends itself to a variety of physical interpretations:

$$v = v_0 \frac{(b - a[r_0/r])^\gamma}{(b-a)^\gamma} . \tag{20}$$

The constants a, b, and γ represent arbitrary dimensionless parameters. We have included extra constants to facilitate substitution later on. Differentiating equation (20), we get

$$\left(\frac{r_0}{r}\right)^2 d\left(\frac{r}{r_0}\right) = \frac{b-a}{\gamma a} \left(\frac{v}{v_0}\right)^{(1-\gamma)/\gamma} d\left(\frac{v}{v_0}\right) . \tag{21}$$

In terms of the variable v, equation (16) becomes

$$E_l dl = \frac{C r_0^3 c}{v_0} \frac{(b-a)}{\gamma a} \int\int \rho_0^2 \left(\frac{v}{v_0}\right)^{(1-\gamma)/\gamma+\beta-3} d\left(\frac{v}{v_0}\right) d\phi dl . \tag{22}$$

At first sight this integrand appears to be independent of the basic variable l. However, the afore-mentioned stepwise character of the function is all-important. With ρ_0 constant, each velocity interval dv corresponds to a given interval dr, wherein we can consider the velocity constant. A given element of velocity v contributes to the integral only over the half-range,

$$|l| = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v}{c}. \tag{23}$$

Thus, if our shell has minimum and maximum velocities, v_1 and v_2, the integral is the limit of a sum, as indicated schematically in Figure 1. The line is flat-topped from $l = 0$ to $l_1 = v_1/c$. Thereafter the intensity declines to zero at $l_2 = v_2/c$. The only values of

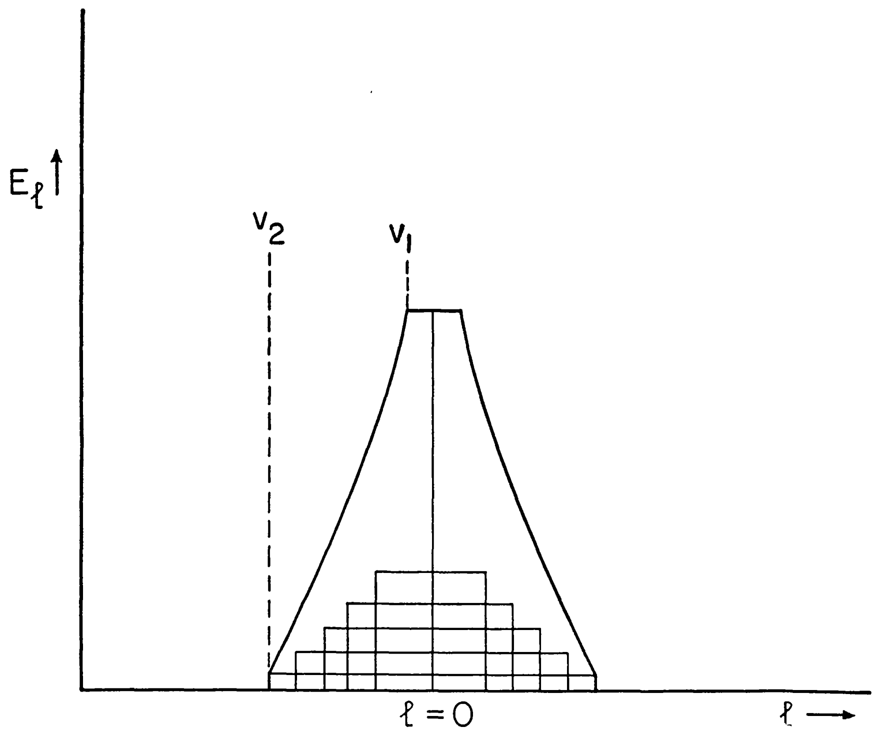

FIG. 1.—Theoretical line profiles for the values of the parameters indicated

v that contribute to the intensity at a given l are those for which $v \geq cl$, when $l > v_1/c$. The integral, therefore, has two sets of limits. Over the range $-l \leq v_1/c \leq l$, we integrate from v_1 to v_2, and the result is independent of l. Over the range $v_2/c \geq |l| \geq v_1/c$, we integrate from $v = cl$ to v_2. Thus

$$E_l dl = 2\pi C r_0^3 \rho_0^2 \frac{(b-a)}{\gamma a} \left(\frac{c}{v_0}\right)^{a+1} \frac{l_2^a - l_1^a}{a} dl \qquad (|l| \leqslant l_1);$$

$$E_l dl = 2\pi C r_0^3 \rho_0^2 \frac{(b-a)}{\gamma a} \left(\frac{c}{v_0}\right)^{a+1} \frac{l_2^a - l^a}{a} dl \qquad (l_2 \geqslant |l| \geqslant l_1), \tag{24}$$

where

$$a = \frac{1-\gamma}{\gamma} + \beta - 2. \tag{25}$$

III. NONUNIFORM DISTRIBUTION OF EMISSION

Before discussing the physical consequences of these formulae, let us consider one further general case. Suppose that the emission is not uniform over the surface of the sphere. Assume, for example, that

$$\rho_0^2 = \rho_1^2 \cos^2 \theta' , \tag{26}$$

where θ' is the angle measured from the pole. In other words, we assume that the emission is greatest near the poles, from which point it declines to zero at the equator. The emission thus occupies a dumbbell-shaped volume surrounding each pole. We further

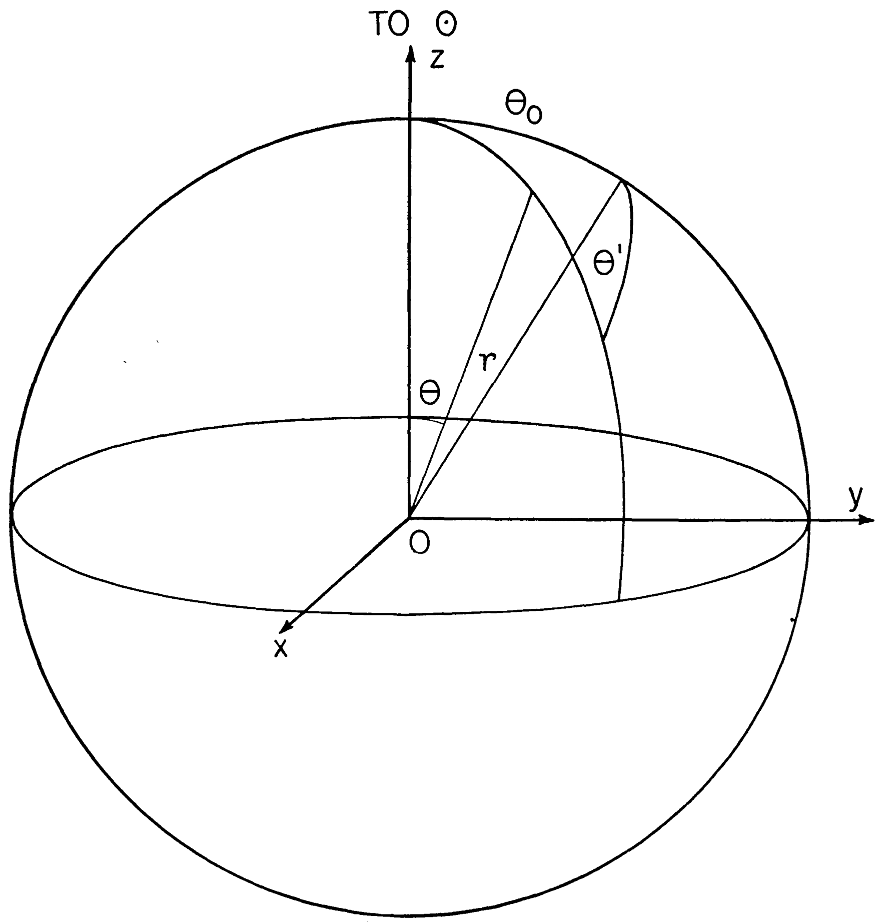

FIG. 2.—The co-ordinate system

take the axis inclined at an angle θ_0 to the observer's line of sight, as shown in Figure 2. By spherical trigonometry,

$$\cos \theta' = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos \phi , \tag{27}$$

which equation enables us to express cos θ' in the co-ordinate system of the observer. We substitute from equations (26) and (27) in equation (22) and integrate first with respect to φ. As

$$\int_0^{2\pi} \cos^2 \theta' d\phi = \pi \left[(3 \cos^2 \theta_0 - 1) \cos^2 \theta + \sin^2 \theta_0 \right] ,$$

equation (24) becomes

$$E_l' dl = \frac{\pi C r_0^3 c}{v_0} \frac{(b-a)}{\gamma a} \rho_1^2 \int \left[(3\cos^2\theta_0 - 1)\, l^2 \frac{c^2}{v_0^2} \left(\frac{v_0}{v}\right)^2 + \sin^2\theta_0 \right] \left(\frac{v}{v_0}\right)^{a-1} d\left(\frac{v}{v_0}\right) dl\,. \tag{28}$$

The limits of integration stand as in the earlier problem. Hence

$$E_l' dl = \pi C r_0^3 \rho_1^2 \left(\frac{c}{v_0}\right)^{a+1} \frac{(b-a)}{\gamma a} \left[\frac{l_2^{a-2} - l_1^{a-2}}{a-2}\, l^2 (3\cos^2\theta_0 - 1) + \frac{l_2^a - l_1^a}{a} \sin^2\theta_0 \right] \quad (|l| \leqslant l_1)\,;$$

$$E_l' dl = \pi C r_0^3 \rho_1^2 \left(\frac{c}{v_0}\right)^{a+1} \frac{(b-a)}{\gamma a} \left[\frac{l_2^{a-2} - l^{a-2}}{a-2}\, l^2 (3\cos^2\theta_0 - 1) + \frac{l_2^a - l^a}{a} \sin^2\theta_0 \right] \quad (|l| \geqslant l_1)\,. \tag{29}$$

The symbol E_l' refers to the emission in the dumbbell-shaped region around the two poles. As an alternative, if the emission is greatest at the equator and zero at the poles, we can write, instead of equation (26),

$$\rho_0^2 = \rho_1^2 \sin^2\theta\,. \tag{30}$$

Denoting this emission by E_l'', we get

$$E_l'' dl = (E_l - E_l')\, dl\,, \tag{31}$$

with ρ_1 substituted for ρ_0 in equation (24). This emission occupies a doughnut-shaped volume around the star's equator.

IV. CHANDRASEKHAR'S MECHANISMS *A* AND *B*

Among the numerous possibilities governing the over-all dynamical situation, Chandrasekhar considers two main types of ejection which he calls mechanisms *A* and *B*. In the first mechanism the atoms at the boundary of the star are repelled by some kind of force that is proportional to the gravitational attraction. In mechanism *B* the atom at the boundary of a star might receive a large initial velocity and, in escaping from the star, be continually decelerated in the gravitational field of the star. The atom either may escape from the star with a finite outward velocity or may fall back after ascending a certain distance.

The postulate of mechanism *A* depended much on Milne's theories of radiation pressure. As is now well known, radiation pressure cannot exert such large forces as would be necessary to produce such large-scale ejection, and hence mechanism *A* acquires an artificial aspect if radiation pressure is to play a dominant role. However, it would be of theoretical interest to consider the effects of a velocity law as postulated in mechanism *A*.

We are now prepared to discuss certain of the cases in which the physical conditions fix the parameters. The *vis viva* integral for matter moving in an inverse-square field gives

$$v = v_0 \left(\frac{b - a\,[r_0/r]}{b-a} \right)^{1/2}. \tag{32}$$

This equation postulates the existence of a repulsive force so that the velocity increases outward. If we take $a = b$, so that $v_0 = 0$ at $r = r_0$, the solution degenerates to the second equation of equations (24). Since $\gamma = \frac{1}{2}$, we have $a = -1$ when $\beta = 0$. Then

$$E_l dl \alpha\, \rho_0^2 (b-a)(l^{-1} - l_2^{-1})\, dl \tag{33}$$

with

$$l_2 = \frac{v_\infty}{c},$$

where v_∞ is the terminal velocity. Here, as Chandrasekhar has noted, we encounter several problems. The first is that $(b - a)$ vanishes by hypothesis. The second is that ρ_0 becomes infinite at r_0, because $v_0 = 0$. Third, we note that $l^{-1} \rightarrow \infty$ as $l \rightarrow 0$, so that the line attains infinite central intensity. The first two of these problems we can overcome by making $\rho_0^2(b - a) = \infty \cdot 0$ approach a finite limit. The third difficulty, however, is serious. After all, according to our analysis, we should expect infinite intensity from a thin stationary layer of infinite density. The difficulty is formal, of course, and lies in the character of the mathematical approximation.

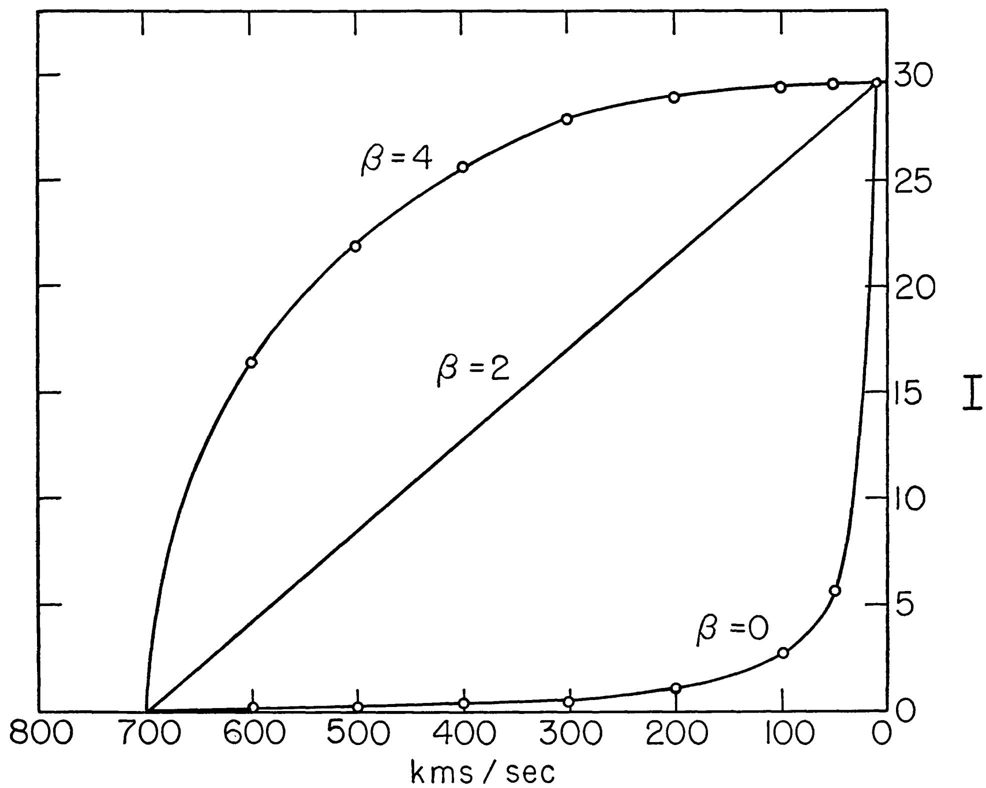

FIG. 3.—Theoretical line profiles for the values of the parameters indicated

To obtain a solution, we may adopt a nonzero value of β, as does Chandrasekhar, or we can start the solution at some finite v_0 and get a finite, flat-topped profile in place of the infinite, peaked one. Also, since α determines the form of the profile, we have the choice of altering γ or selecting a nonzero β. If we adopt the former procedure, we obtain an arbitrary velocity law corresponding to an arbitrary and noninverse-square field.

The case of mechanism B requires that the atoms move outward with a large starting velocity. Thereafter, they are subject to deceleration. For an inverse-square field we take

$$v = v_0 \left(\frac{b + a\,(r_0/r)}{b + a} \right)^{1/2},$$

where a is positive. Then the character of the solution depends on the sign of b. If b is positive, the envelope extends to infinity with a finite terminal velocity. If b is nega-

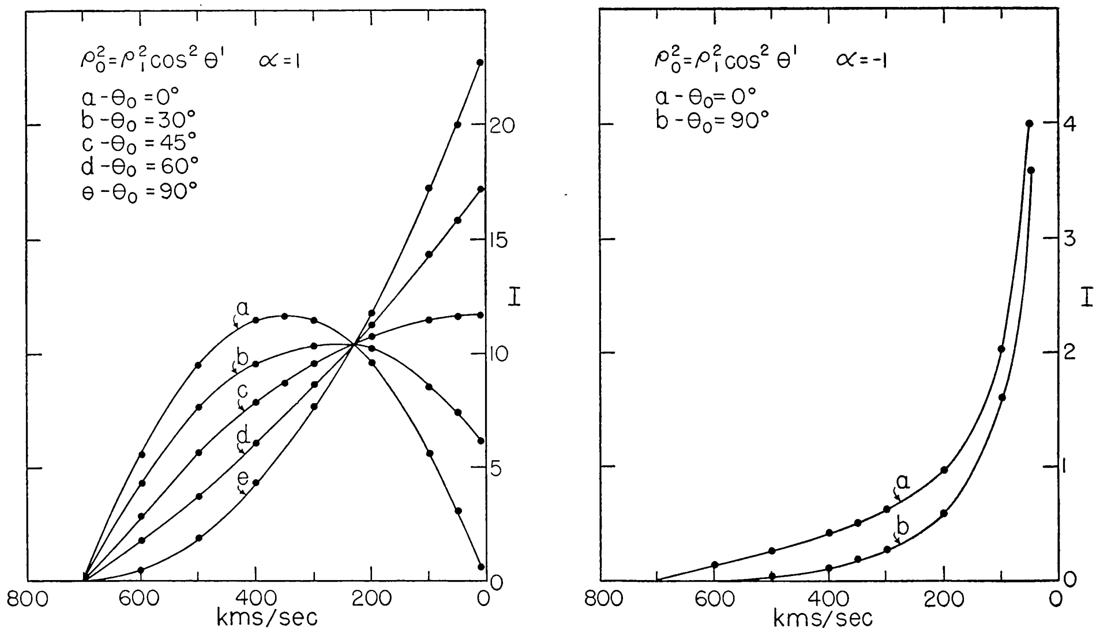

FIG. 4.—Theoretical line profiles for the values of the parameters indicated

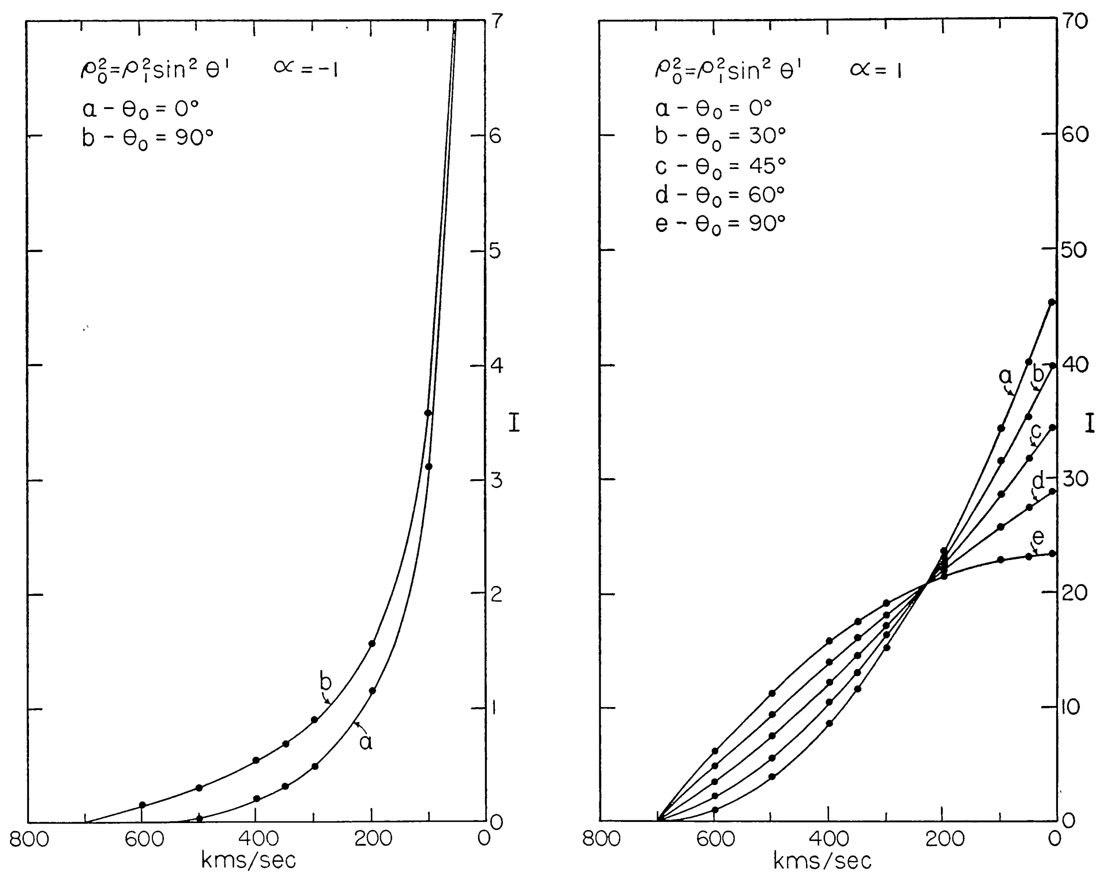

FIG. 5.—Theoretical line profiles for the values of the parameters indicated

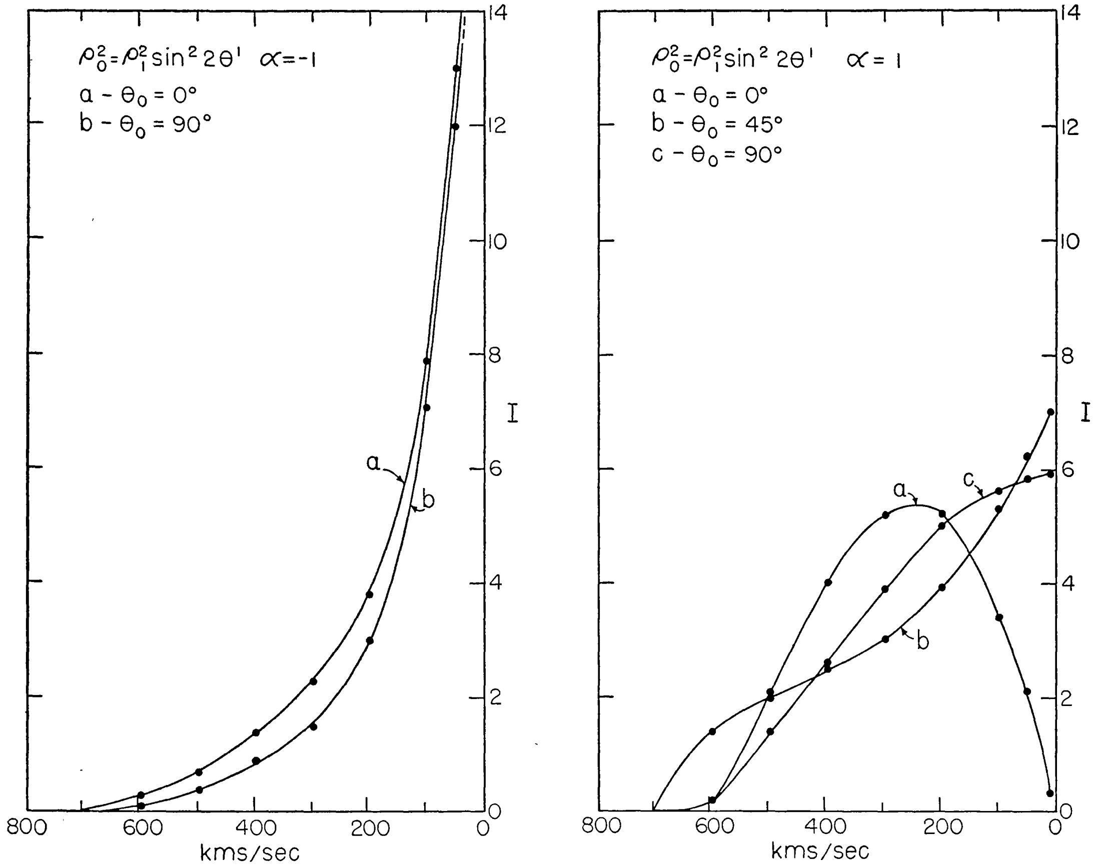

FIG. 6.—Theoretical line profiles for the values of the parameters indicated

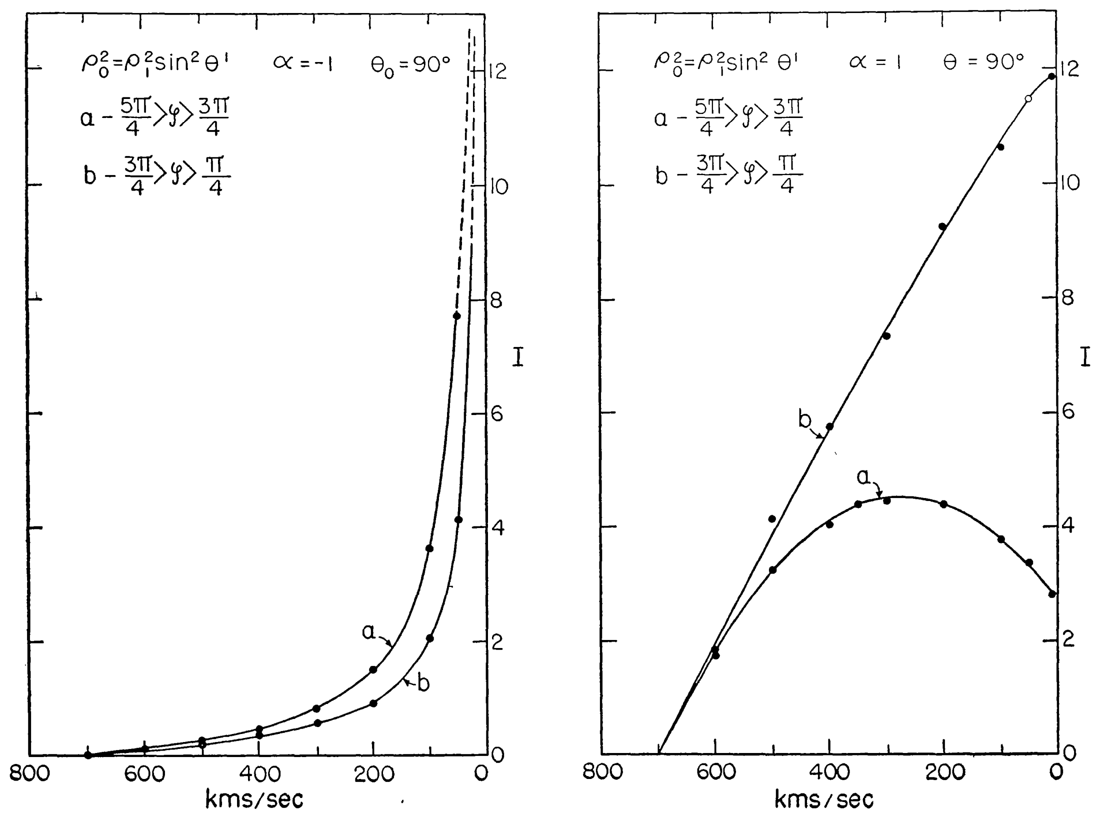

FIG. 7.—Theoretical line profiles for the values of the parameters indicated

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tive, the velocity falls to zero at the finite distance $r = -ar_0/b$. Whenever $v \to 0$, indeterminacy arises as before, because ρ_0 must then be infinite.

Figure 3 demonstrates the change in form of the line contour for different values of β. The profiles result from the assumption that the emission envelope is spherically symmetrical and that the limiting velocities v_2 and v_1 are 700 and 10 km/sec, respectively. With these same limiting values of velocity, the profiles calculated for cases of nonuniform distribution of emission appear in Figures 4, 5, 6, and 7.

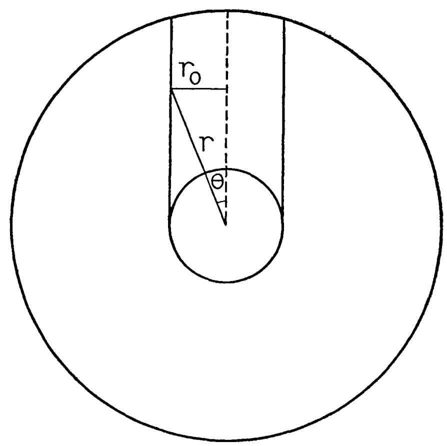

FIG. 8.—The occultation effect

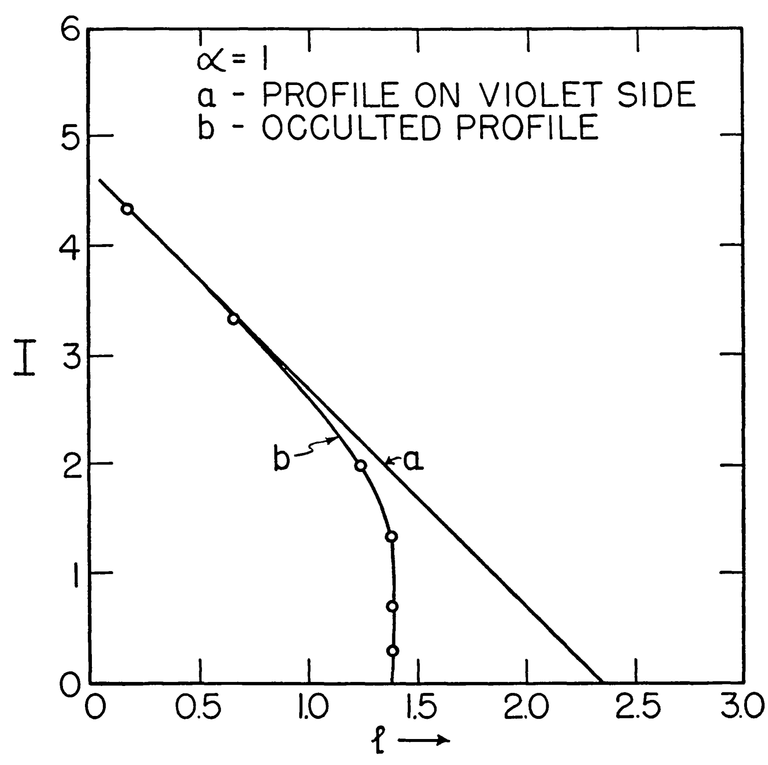

FIG. 9.—The influence of the occultation effect on the line profiles

V. THE OCCULTATION EFFECT

The co-ordinate system and the order of integration here used lend themselves naturally to the consideration of the occultation effect. The geometry of the situation is shown in Figure 8. The star occults the portion of the shell that lies behind it, this region being defined by a tangent cylinder along the line of sight to the surface of the star. We can take care of this effect by using proper limits for the evaluation of the intensity at any value l. We get

$$1-\frac{u^2}{v^2}=\sin^2\theta=\left(\frac{r_0}{r}\right)^2,$$

where r_0 is the radius of the star and r is the radical distance of a point along the limiting visible boundary. Then the limiting value becomes

$$l_i=\frac{v}{c}\left[1-\left(\frac{r_0}{r}\right)^2\right]^{1/2}. \tag{34}$$

The part played by occultation is then merely to shift the intensities for any value l to its correspondingly modified value, l_i. Figure 9 demonstrates the effect when the motion in the envelope follows mechanism B with the previously considered limiting velocities and the assumption that the radius of the envelope is ten times that of the parent-star.

VI. MISCELLANEOUS COMMENTS

We point out that laws (26) and (30), here used to describe the latitude variation, are specific examples of a more general procedure, viz., expansion of the excitation function in terms of spherical harmonics.

Menzel and Payne[6] noted certain peculiar characteristics observed in the spectrum of Nova Aquilae 1918. They pointed out that the "dipped" and "peaked" profiles observed at different times in the spectral lines of this nova could be explained on the basis of ejection or excitation that was a function of latitude.

[6] *Proc. Nat. Acad. Sci.*, **19**, 641, 1933.