Bull. Astr. Soc. India (1987) 15, 1–12

Transmission and reflection operators of radiative transfer equation with aberration and advection terms. I. Monochromatic radiation field with spherical symmetry

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Received 1986 November 4; accepted 1987 December 15

Abstract. Integration of the radiative transfer equation, with the aberration and advection terms included, is described, assuming coherent and isotropic scattering. The operators of transmission and reflection derived in this paper are applicable to situations in which $v/c = 0.0167$ where v and c are the velocities of the gas and light respectively. These operators can be applied to a spherically symmetric medium in which matter scatters or absorbs and emits monochromatic radiation.

Key words : Radiative transfer—aberration—advection—coherent and isotropic scattering

1. Introduction

The velocities of gases in the outer layers of supergiant stars, novae, supernovae, active galactic nuclei, etc., are so large that the factor v/c, where v and c are the velocities of gas and light respectively, is a substantial fraction. In such situations, the radiation field in the moving gases gets substantially modified after emission and absorption by gas. The effects of aberration and advection caused by high velocities must be investigated in any serious calculation of the radiation field. Mihalas, Kunasz & Hummer (1976) included the aberration and advection terms corresponding to velocities of the order $v/c \approx 0.01$ in the line calculations. They found that these do not produce any observable changes and that it is the Doppler shifts which will change the line radiation. Mihalas & Klein (1982) have pointed out that time-dependent transfer will be of importance.

Recently, we have investigated the effects of the aberration and advection terms of the order $v/c \sim 0.01$. We started the investigation by studying the order of magnitude changes when aberration and advection terms are included (Peraiah 1986). This study revealed that the terms due to aberration and advection are of the same order as, or some times exceed, the usual divergent terms of radiative transfer equation. Therefore it is instructive to calculate the radiation field by solving the radiative transfer equation with these terms included. We have assumed simple model with coherent, isotropic, monochromatic radiation field in a plane parallel medium moving with velocities up to 5000 km s^{-1} *i.e.* $v/c = 0.0167$. To our surprise, we found differences as large as 20% to 90% in a medium whose optical depths change from 5 to 30 or 50 (Peraiah 1987). It is therefore evident that we shall have to seriously consider including these terms in the solution of radiative transfer equation.

In this paper, we derive transmission and reflection operators which lead to the solution of radiative transfer equation with aberration and advection terms included in spherical symmetry.

2. Development of the solution

The equation of radiative transfer in a spherically symmetric media in the comoving frame of the fluid with relativistic terms of the order of v/c is written as (Castor 1972; Mihalas 1978; Munier & Weaver 1986)

$$(\mu^o + \beta)\frac{\partial U(r, \mu^o)}{\partial r} + \frac{1 - \mu^{o2}}{r}\left[1 + \mu^o\beta\left(1 - \frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\frac{\partial U(r, \mu^o)}{\partial \mu^o}$$

$$= K(r)\,[S(r) - U(r, \mu^o)] + \frac{2(\mu^o + \beta)\,U(r, \mu^o)}{r}$$

$$- 3\left[\frac{\beta}{r}(1 - \mu^{o2}) + \mu^{o2}\frac{d\beta}{dr}\right]U(r, \mu^o), \qquad \ldots(1)$$

where $\beta = \frac{v}{c}$, $\mu^o = \frac{\mu - \beta}{1 - \mu\beta}$, $U(r, \mu^o) = 4\pi r^2 I(r, \mu^o)$.

$I(r, \mu^o)$ is the specific intensity of the ray whose direction makes an angle of $\cos^{-1}\mu$ with the radius vector r. $K(r)$ is the absorption coefficient; and $S(r)$ is the source function. Equation (1) can be integrated on the angle-radius mesh defined on $[\mu_{j-1}, \mu_j]$ $[r_{i-1}, r_i]$ (see Peraiah & Varghese 1985). First we shall express the specific intensity in terms of certain interpolation coefficients and weight factors :

$$U(r, \mu^o) = U_o + U_r\xi + U_{\mu^o}\eta + U_{r\mu^o}\xi\eta, \qquad \ldots(2)$$

where U_o, U_r, U_{μ^o}, $U_{r\mu^o}$ are the interpolation coefficients. The source function S is similarly defined. In equation (2)

$$\xi = \frac{r - \bar{r}}{\Delta r/2} \text{ and } \eta = \frac{\mu^o - \bar{\mu}^o}{\Delta\mu^o/2}, \qquad \ldots(3)$$

where

$$\bar{r} = (r_{i-1} + r_i)/2, \quad \Delta r = (r_i - r_{i-1}) \qquad \ldots(4)$$

$$\bar{\mu}^o = (\mu^o_{j-1} + \mu^o_j)/2, \quad \Delta\mu^o = (\mu^o_j - \mu^o_{j-1}). \qquad \ldots(5)$$

From equations (1)–(4), we have

$$(\mu^o + \beta)\,\frac{\partial U(r, \mu^o)}{\partial r} = \frac{2}{\Delta r}\,(\mu^o + \beta)\,(U_r + U_{r\mu^o}\eta). \qquad \text{...(6)}$$

We shall apply the operator

$$X_{\mu^o} = \frac{1}{\Delta\mu^o}\int_{\Delta\mu^o} \ldots d\mu^o \qquad \text{...(7)}$$

on equation (6). This operator is written as

$$\frac{1}{\Delta\mu^o}\int_{\Delta\mu^o} (\mu^o + \beta)\,\frac{\partial U(r, \mu^o)}{\partial r}\,d\mu^o = \frac{1}{\Delta\mu^o}\int_{\Delta\mu^o} \frac{2}{\Delta r}\,(\mu^o + \beta)\,(U_r + U_{r\mu^o}\eta). \qquad \text{...(8)}$$

The effects of the operator X_{μ^o} on various terms on the R.H.S. of equation (8) are as follows :

$$X_{\mu^o}[Z] = Z, \text{ where } Z \text{ is a const}, \qquad \text{...(9)}$$

$$X_{\mu^o}[\mu^o] = \overline{\mu^o}, \qquad \text{...(10)}$$

$$X_{\mu^o}[\eta\mu^o] = \tfrac{1}{6}\,\Delta\mu^o, \qquad \text{...(11)}$$

$$X_{\mu^o}[\eta] = 0, \qquad \text{...(12)}$$

$$X_{\mu^o}[\mu^{o2}\eta] = \tfrac{1}{3}\,\Delta\mu^o\,\overline{\mu^o}, \qquad \text{...(13)}$$

$$X_{\mu^o}[\mu^{o2}] = \overline{\mu^{o2}}. \qquad \text{...(14)}$$

Equation (8) becomes in the light of the identities (10)–(14)

$$X_{\mu^o}\left[(\mu^o + \beta)\,\frac{\partial U}{\partial r}\right] = X_{\mu^o}\left[\frac{2}{\Delta r}\,(\mu^o + \beta)\,(U_r + U_{r\mu^o}\eta)\right]$$

$$= \frac{2}{\Delta r}\,\{(\mu^o + \beta)\,U_r + \tfrac{1}{6}\,\Delta\mu^o U_{r\mu^o}\}. \qquad \text{...(15)}$$

The second term on L.H.S. of equation (1) can be written as

$$\frac{1-\mu^{o2}}{r}\left[1 + \mu^o\beta\left(1 - \frac{r}{\beta}\,\frac{d\beta}{dr}\right)\right]\frac{\partial U(r, \mu^o)}{\partial\mu^o}$$

$$= \frac{1-\mu^{o2}}{r}\left[1 + \mu^o\beta\left(1 - \frac{r}{\beta}\,\frac{d\beta}{dr}\right)\right]\frac{\partial}{\partial\mu^o}\,(U_o + U_r\xi + U_{\mu^o}\eta + U_{r\mu^o}\xi\eta), \qquad \text{...(16)}$$

where we have used equation (2). Equation (16) reduces to

$$\frac{1-\mu^{o2}}{r}\left[1 + \mu^o\beta\left(1 - \frac{r}{\beta}\,\frac{d\beta}{dr}\right)\right]\frac{\partial U}{\partial\mu^o}$$

$$= \frac{1-\mu^{o2}}{r}\left[1 + \mu^o\beta\left(1 - \frac{r}{\beta}\,\frac{d\beta}{dr}\right)\right]\frac{2}{\Delta\mu^o}\,(U_{\mu^o} + U_{r\mu^o}\xi). \qquad \text{...(17)}$$

We shall consider now the terms containing U on the R.H.S. of equation (1). By using equation (2), we have

$$-\left[K + \frac{2}{r}(\mu^o + \beta) - 3\left\{\frac{\beta}{r}(1 - \mu^{o2}) + \mu^{o2}\frac{d\beta}{dr}\right\}\right]U$$

$$= -\left[K + \frac{1}{r}\{2(\mu^o + \beta) - 3\beta(1 - \mu^{o2})\} - 3\mu^{o2}\frac{d\beta}{dr}\right]$$

$$\times [U_o + U_r\xi + U_{\mu^o}\eta + U_{r\mu^o}\xi\eta]. \quad \ldots(18)$$

Applying X_{μ^o} on equations (6), (17) and (18), we obtain

$$\frac{2}{\Delta r}\{(\bar{\mu}^o + \beta)U_r + \tfrac{1}{6}\Delta\mu^o U_{r\mu^o}\} + \frac{2}{r.\Delta\mu^o}\{(1 - \overline{\mu^{o2}})$$

$$+ \bar{\mu}^o(1 - \langle\mu^{o\,2}\rangle)\beta R\}(U_{\mu^o} + U_{r\mu^o}\xi)$$

$$= K[(S_o + S_r\xi) - (U_o + U_r\xi)] + \left\{\frac{2\bar{\mu}^o + \beta(3\overline{\mu^{o2}} - 1)}{r} - 3\overline{\mu^{o2}}\frac{d\beta}{dr}\right\}$$

$$\times (U_o + U_r\xi) + \Delta\mu^o\bar{\mu}^o\left\{\frac{1}{r}\left(\beta + \frac{1}{3\bar{\mu}^o}\right) - \frac{d\beta}{dr}\right\}(U_{\mu^o} + U_{r\mu^o}\xi), \quad \ldots(19)$$

where $$R = 1 - \frac{r}{\beta}\frac{d\beta}{dr}. \quad \ldots(20)$$

We shall keep $d\beta/dr$ constant over a small increment of the radius or in a shell whose optical thickness does not exceed the critical stepsize :

$$\frac{d\beta}{dr} = \text{const} = s. \quad \ldots(21)$$

Then $$\Delta\beta = s\cdot\Delta r, \quad \ldots(22)$$

where $$\Delta r = (r_i - r_{i-1}), \quad \ldots(23)$$

$$\langle\mu^{o2}\rangle = \tfrac{1}{2}(\mu_j^{o2} + \mu_{j-1}^{o2}). \quad \ldots(24)$$

To perform the radias integration, we apply the operator

$$Y_V = \frac{1}{V}\int_{\Delta r}\ldots\ldots 4\pi r^2 dr, \quad \ldots(25)$$

where $$V = \frac{4\pi}{3}(r_i^3 - r_{i-1}^3), \quad \ldots(26)$$

on equation (19) and obtain

$$\frac{2}{\Delta r}\{(\bar{\mu}^o + \beta)U_r + \tfrac{1}{6}\Delta\mu^o U_{r\mu^o}\} + \left\{\tfrac{1}{2}\frac{\Delta A}{V}\cdot G - \frac{G}{\beta}\frac{d\beta}{dr}\right.$$

$$\left. + \frac{\Delta A}{V}\frac{1 - \bar{\mu}^o}{\Delta\mu^o}\right\}U_{\mu^o} + \left\{\frac{G}{\Delta r}\left(2 - \frac{r\Delta A}{V}\right) - \right.$$

equation continued

$$- \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \frac{G}{\beta} \frac{d\beta}{dr} + \frac{2}{\Delta r} \frac{1 - \overline{\mu^{o2}}}{\Delta \mu^o} \left(2 - \frac{\bar{r}\Delta A}{V}\right)\Big\} U_{r\mu^o}$$

$$= K \left\{\left(S_o + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} S_r\right) - \left(U_o + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} U_r\right)\right\}$$

$$+ \{2\bar{\mu^o} + \beta(3\overline{\mu^{o2}} - 1)\} \left\{\tfrac{1}{2} \frac{\Delta A}{V} U_o + \frac{1}{\Delta r}\left(2 - \frac{\bar{r}\Delta A}{V}\right) U_r\right\}$$

$$- 3\overline{\mu^{o2}} \frac{d\beta}{dr} \left(U_o + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} U_r\right) + \Delta\mu_o \cdot \bar{\mu^o} \left[\left(\beta + \frac{1}{3\bar{\mu^o}}\right) \left\{\tfrac{1}{2} \frac{\Delta A}{V} U_{\mu^o}\right.\right.$$

$$\left.\left. + \frac{1}{\Delta r} \left(2 - \frac{\bar{r} \cdot \Delta A}{V}\right) U_{r\mu^o}\right\} - \frac{d\beta}{dr} \left(U_{\mu^o} + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} U_{r\mu^o}\right)\right], \qquad \ldots(27)$$

where $$G = \frac{2\bar{\mu^o}}{\Delta\mu^o} (1 - \langle\mu^{o2}\rangle)\, \beta, \qquad \ldots(28)$$

$$\Delta A = 4\pi(r_i^2 - r_{i-1}^2), \qquad \ldots(29)$$

$$\bar{A} = V/\Delta r. \qquad \ldots(30)$$

In the plane parallel case equation (27) reduces to

$$\frac{2}{\Delta r} \{(\bar{\mu^o} + \beta)\, U_r + \tfrac{1}{6} \Delta\mu^o U_{r\mu^o}\}$$

$$= K(S_o - U_o) - \frac{d\beta}{dr} \left\{3\overline{\mu^{o2}} U_o + \Delta\mu^o \cdot \bar{\mu^o} U_{\mu^o} - \frac{G}{\beta} \frac{d\beta}{dr} U_{\mu^o}\right\}. \qquad \ldots(31)$$

Collecting the coefficients of the interpolation coefficients U_o, U_r, etc., we can rewrite equation (27)

$$U_r \left[\frac{2}{\Delta r} (\bar{\mu^o} + \beta) + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} K - P\{2\bar{\mu^o} + \beta(3\overline{\mu^{o2}} - 1)\} + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} 3\overline{\mu^{o2}} \frac{d\beta}{dr}\right]$$

$$+ U_{\mu^o} \left[G\left(\tfrac{1}{2} \frac{\Delta A}{V} - \frac{1}{\beta} \frac{d\beta}{dr}\right) + \frac{\Delta A}{V} \left(\frac{1 - \overline{\mu^{o2}}}{\Delta\mu^o}\right)\right.$$

$$\left. - \Delta\mu^o \cdot \bar{\mu^o} \left\{\left(\beta + \frac{1}{3\bar{\mu^o}}\right) \tfrac{1}{2} \frac{\Delta A}{V} - \frac{d\beta}{dr}\right\}\right] + U_{r\mu^o} \left[\tfrac{1}{3} \frac{\Delta\mu^o}{\Delta r}\right.$$

$$+ Gp - \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \frac{G}{\beta} \frac{d\beta}{dr} + 2p \frac{1 - \overline{\mu^{o2}}}{\Delta\mu^o} - \Delta\mu^o \cdot \bar{\mu^o} \left\{\left(\beta + \frac{1}{3\bar{\mu^o}}\right) p\right.$$

$$\left.\left. - \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \frac{d\beta}{dr}\right\}\right] + U_o\left[K - \tfrac{1}{2} \frac{\Delta A}{V} \{2\bar{\mu^o} + \beta(3\overline{\mu^{o2}} - 1)\} + 3\overline{\mu^{o2}} \frac{d\beta}{dr}\right]$$

$$= K\left[S_o + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} S_r\right]. \qquad \ldots(32)$$

For $-\mu^{\circ}$, the transfer equation is written as

$$(-\mu^{\circ}+\beta)\frac{\partial U(r,-\mu^{\circ})}{\partial r}-\frac{1-\mu^{\circ 2}}{r}\left[1-\mu^{\circ}\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right]\frac{\partial U(r,-\mu^{\circ})}{\partial\mu^{\circ}}$$

$$=K[S(r)-U(r,-\mu^{\circ})]+2\frac{(-\mu^{\circ}+\beta)}{r}U(r,-\mu^{\circ})$$

$$-3\left[\frac{\beta}{r}(1-\mu^{\circ 2})+\mu^{\circ 2}\frac{d\beta}{dr}\right]U(r,-\mu^{\circ}). \qquad \ldots(33)$$

Substituting equation (2) in equation (33), we obtain

$$\frac{2}{\Delta r}(-\mu^{\circ}+\beta)(U_r+U_{r\mu^{\circ}}\eta)-\frac{2}{\Delta\mu^{\circ}}\left(\frac{1-\mu^{\circ 2}}{r}\right)\left[1-\mu^{\circ}\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right]$$

$$\times(U_{\mu^{\circ}}+U_{r\mu^{\circ}}\xi)=K(r)S(r)+\left[K(r)+\frac{1}{r}\{2(-\mu^{\circ}+\beta)\right.$$

$$\left.-3\beta(1-\mu^{\circ 2})\}-3\mu^{\circ 2}\frac{d\beta}{dr}\right](U_{\circ}+U_r\xi+U_{\mu^{\circ}}\eta+U_{r\mu^{\circ}}\xi\eta). \qquad \ldots(34)$$

Application of $X_{\mu^{\circ}}$ on equation (34) gives

$$\frac{2}{\Delta r}\{(\beta-\bar{\mu}^{\circ})U_r-\tfrac{1}{6}\Delta\mu^{\circ}U_{r\mu^{\circ}}\}-\frac{2}{\Delta\mu^{\circ}}\cdot\frac{1}{r}\left\{(1-\overline{\mu^{\circ 2}})\right.$$

$$\left.-\bar{\mu}^{\circ}(1-\langle\overline{\mu^{\circ 2}}\rangle)\beta\left(1-\frac{r}{\beta}\frac{d\beta}{dr}\right)\right\}(U_{\mu^{\circ}}+U_{r\mu^{\circ}}\xi)$$

$$=K[(S_{\circ}+S_r\xi)-(U_{\circ}+U_r\xi)]-\left\{\frac{2\mu^{\circ}+\beta(1-3\overline{\mu^{\circ 2}})}{r}\right.$$

$$\left.+3\overline{\mu^{\circ 2}}\frac{d\beta}{dr}\right\}(U_{\circ}+U_r\xi)-\frac{\Delta\mu^{\circ}\cdot\mu^{\circ}}{r}\left(r\frac{d\beta}{dr}-\beta+\frac{1}{3\bar{\mu}^{\circ}}\right)$$

$$\times(U_{\mu^{\circ}}+U_{r\mu^{\circ}}\xi). \qquad \ldots(35)$$

Applying the operator Y_v on equation (35) we obtain

$$\frac{2}{\Delta r}\{(\beta-\bar{\mu}^{\circ})U_r-\tfrac{1}{6}\Delta\mu^{\circ}U_{r\mu^{\circ}}\}+\left(\tfrac{1}{2}\frac{\Delta A}{V}G-\frac{G}{\beta}\frac{d\beta}{dr}-\frac{\Delta A}{V}\frac{1-\overline{\mu^{\circ 2}}}{\Delta\mu^{\circ}}\right)$$

$$\times U_{\mu^{\circ}}+\left\{Gp-\tfrac{1}{6}\frac{\Delta A}{A}\frac{G}{\beta}\frac{d\beta}{dr}-\frac{2}{\Delta\mu^{\circ}}(1-\overline{\mu^{\circ 2}})p\right\}U_{r\mu^{\circ}}$$

$$=K\left\{\left(S_{\circ}+\tfrac{1}{6}\frac{\Delta A}{A}S_r\right)-\left(U_{\circ}+\tfrac{1}{6}\frac{\Delta A}{A}U_r\right)\right\}$$

$$-\left\{\tfrac{1}{2}\frac{\Delta A}{V}(2\bar{\mu}^{\circ}+\beta-3\beta\overline{\mu^{\circ 2}})+3\overline{\mu^{\circ 2}}\frac{d\beta}{dr}\right\}U_{\circ}$$

$$-\left\{p(2\bar{\mu}^{\circ}+\beta-3\beta\overline{\mu^{\circ 2}})+3\overline{\mu^{\circ 2}}\frac{d\beta}{dr}\cdot\tfrac{1}{6}\frac{\Delta A}{A}\right\}U_r-$$

equation continued

$$-\left\{\Delta\mu^o\cdot\bar{\mu}^o\left[\frac{d\beta}{dr}+\tfrac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\mu^o}-\beta\right)\right]\right\}U_{\mu^o}$$

$$-\left\{\Delta\mu^o\cdot\bar{\mu}^o\left[\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}+p\left(\frac{1}{3\overline{\mu^o}}-\beta\right)\right]\right\}U_{r\mu^o}, \quad ...(36)$$

where $$p=\frac{1}{\Delta r}\left(2-\frac{\bar{r}\cdot\Delta A}{V}\right). \quad ...(37)$$

Equation (37) is rewritten after rearranging the interpolation coefficients :

$$U_r\left[\frac{2}{\Delta r}(\beta-\bar{\mu}^o)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}K+p(2\bar{\mu}^o+\beta-3\beta\overline{\mu^{o2}})+3\overline{\mu^{o2}}\frac{d\beta}{dr}\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right]$$

$$+U_{\mu^o}\left[\tfrac{1}{2}\frac{\Delta A}{V}G-\frac{G}{\beta}\frac{d\beta}{dr}-\frac{\Delta A}{V}\frac{1-\overline{\mu^{o2}}}{\Delta\mu^o}+\Delta\mu^o\bar{\mu}^o\left\{\frac{d\beta}{dr}\right.\right.$$

$$\left.\left.+\tfrac{1}{2}\frac{\Delta A}{V}\left(\frac{1}{3\overline{\mu^o}}-\beta\right)\right\}\right]+U_{r\mu^o}\left[-\tfrac{1}{3}\frac{\Delta\mu^o}{\Delta r}+Gp-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{\beta}\frac{d\beta}{dr}\right.$$

$$\left.-\frac{2}{\Delta\mu^o}(1-\overline{\mu^{o2}})p+\Delta\mu^o\cdot\bar{\mu}^o\left\{\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{d\beta}{dr}+p\left(\frac{1}{3\overline{\mu^o}}-\beta\right)\right\}\right]$$

$$+U_o\left[K+\tfrac{1}{2}\frac{\Delta A}{V}(2\bar{\mu}^o+\beta-3\beta\overline{\mu^{o2}}+3\overline{\mu^{o2}}\frac{d\beta}{dr}\right]$$

$$=K\left(S_o+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right). \quad ...(38)$$

Multiplying equations (32) and (38) by Δr we get

$$\alpha U_r+\beta U_{\mu^o}+\gamma U_{r\mu^o}+\delta U_o=\Delta r\cdot K\left[S_o+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right] \quad ...(39)$$

and $$\alpha' U_r+\beta' U_{\mu^o}+\gamma' U_{r\mu^o}+\delta' U_o=\Delta r\cdot K\left[S_o+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}S_r\right], \quad ...(40)$$

where $$\alpha=\Delta r\left\{\frac{2}{\Delta r}(\bar{\mu}^o+g)+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}K-p[2\bar{\mu}^o+g(3\overline{\mu^{o2}}-1)\right.$$

$$\left.+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}3\overline{\mu^{o2}}\frac{dg}{dr}\right\}, \quad ...(41)$$

$$\beta=\Delta r\left\{G\left(\tfrac{1}{2}\frac{\Delta A}{V}-\frac{1}{g}\frac{dg}{dr}\right)+\frac{\Delta A}{V}\left(\frac{1-\overline{\mu^{o2}}}{\Delta\mu^o}\right)\right.$$

$$\left.-\Delta\mu^o\bar{\mu}^o\left[\left(g+\frac{1}{3\mu^o}\right)\tfrac{1}{2}\frac{\Delta A}{V}-\frac{dg}{dr}\right]\right\}, \quad ...(42)$$

$$\gamma=\Delta r\left\{\tfrac{1}{3}\frac{\Delta\mu^o}{\Delta r}+Gp-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{G}{g}\frac{dg}{dr}+2p\frac{1-\overline{\mu^{o2}}}{\Delta\mu^o}\right.$$

$$\left.-\Delta\mu^o\bar{\mu}^o\left[\left(g+\frac{1}{3\overline{\mu^o}}\right)p-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\frac{dg}{dr}\right]\right\}, \quad ...(43)$$

$$\delta = \Delta r \left\{ K - \tfrac{1}{2} \frac{\Delta A}{V} [2\bar{\mu}^o + g(3\overline{\mu^{o2}} - 1)] + 3\overline{\mu^{o2}} \frac{dg}{dr} \right\}, \quad \ldots(44)$$

and $$g = v/c. \quad \ldots(45)$$

We have replaced β by g as β is newly defined in equation (42). Furthermore we have,

$$\alpha' = \Delta r \left[\frac{2}{\Delta r} (g - \bar{\mu}^o) + \tfrac{1}{6} \frac{\Delta A}{\bar{A}} K + A_2 \right], \quad \ldots(46)$$

$$\beta' = \Delta r(B_1 + A_3), \quad \ldots(47)$$

$$\gamma' = \Delta r \left(A_4 + B_2 - \tfrac{1}{3} \frac{\Delta \mu^o}{\Delta r} \right), \quad \ldots(48)$$

where $$\delta' = \Delta r(K + A_1), \quad \ldots(49)$$

$$A_1 = \tfrac{1}{2} \frac{\Delta A}{V} (2\bar{\mu}^o + g - 3g\overline{\mu^{o2}}) + 3\overline{\mu^{o2}} \frac{dg}{dr}, \quad \ldots(50)$$

$$A_2 = p(2\bar{\mu}^o + g - 3g\overline{\mu^{o2}}) + 3\overline{\mu^{o2}} \frac{dg}{dr} \cdot \tfrac{1}{6} \frac{\Delta A}{\bar{A}}, \quad \ldots(51)$$

$$A_3 = \Delta\bar{\mu}^o \cdot \bar{\mu}^o \left[\frac{dg}{dr} + \tfrac{1}{2} \frac{\Delta A}{V} \left(\frac{1}{3\bar{\mu}^o} - g \right) \right], \quad \ldots(52)$$

$$A_4 = \Delta\mu^o \cdot \bar{\mu}^o \left[\tfrac{1}{6} \frac{\Delta A}{\bar{A}} \frac{dg}{dr} + p \left(\frac{1}{3\bar{\mu}^o} - g \right) \right], \quad \ldots(53)$$

$$B_1 = \tfrac{1}{2} \frac{\Delta A}{V} G - \frac{G}{g} \frac{dg}{dr} - \frac{\Delta A}{V} \frac{1 - \overline{\mu^{o2}}}{\Delta \mu^o}, \quad \ldots(54)$$

$$B_2 = Gp - \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \frac{G}{g} \frac{dg}{dr} - \frac{2}{\Delta \mu^o} (1 - \overline{\mu^{o2}})\, p. \quad \ldots(55)$$

We now replace U_o, U_r, U_{μ^o}, $U_{r\mu^o}$ by the nodal values U_a, U_b etc. (see Peraiah & Varghese 1985) :

$$\alpha(-U_a - U_b - U_c - U_d) + \beta(-U_a + U_b + U_c - U_d)$$
$$+ \gamma(U_a - U_b - U_c + U_d) + \delta(U_a + U_b + U_c + U_d)$$
$$= \tau \left[(S_a + S_b + S_c + S_d) \left(1 - \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \right) \right] \quad \ldots(56)$$

and $$\alpha'(-U_a - U_b - U_c - U_d) + \beta'(-U_a + U_b + U_c - U_d)$$
$$+ \gamma(U_a - U_b - U_c + U_d) + \delta(U_a + U_b + U_c + U_d)$$
$$= \tau \left(1 - \tfrac{1}{6} \frac{\Delta A}{\bar{A}} \right) (S_a + S_b + S_c + S_d). \quad \ldots(57)$$

Equations (56) and (57) can be rewritten as

$$U_a(-\alpha-\beta+\gamma+\delta)+U_b(-\alpha+\beta-\gamma+\delta)$$
$$+U_c(-\alpha+\beta-\gamma+\delta)+U_d(-\alpha-\beta+\gamma+\delta)$$
$$=\left(1-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a+S_b+S_c+S_d) \qquad \ldots(58)$$

and

$$U_a(-\alpha'-\beta'+\gamma'+\delta')+U_b(-\alpha'+\beta'-\gamma'+\delta')$$
$$+U_c(-\alpha'+\beta'-\gamma'+\delta')+U_d(-\alpha'-\beta'+\gamma'+\delta')$$
$$=\left(1-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a+S_b+S_c+S_d). \qquad \ldots(59)$$

Letting

$$A_a=-\alpha-\beta+\gamma+\delta, \qquad \ldots(60)$$
$$A_b=-\alpha+\beta-\gamma+\delta, \qquad \ldots(61)$$
$$A_c=-\alpha+\beta-\gamma+\delta, \qquad \ldots(62)$$
$$A_d=-\alpha-\beta+\gamma+\delta; \qquad \ldots(63)$$

and

$$A'_a=-\alpha'-\beta'+\gamma'+\delta', \qquad \ldots(64)$$
$$A'_b=-\alpha'+\beta'-\gamma'+\delta', \qquad \ldots(65)$$
$$A'_c=-\alpha'+\beta'-\gamma'+\delta', \qquad \ldots(66)$$
$$A'_d=-\alpha'-\beta'+\gamma'+\delta', \qquad \ldots(67)$$

we can write equations (58) and (59) as

$$A_aU_a^++A_bU_b^++A_cU_c^++A_dU_d^+$$
$$=\left(1-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a^++S_b^++S_c^++S_d^+) \qquad \ldots(68)$$

and

$$A'_aU_a^-+A'_bU_b^-+A'_cU_c^-+A'_dU_d^-$$
$$=\left(1-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_a^-+S_b^-+S_c^-+S_d^-). \qquad \ldots(69)$$

By writing the nodal values in their full form such as

$$U_a^+=U_{j-1}^{i-1,+},\quad U_b^+=U_j^{i-1,+}, \text{ etc.}; \qquad \ldots(70)$$

we obtain

$$A_aU_{j-1}^{i-1,+}+A_bU_j^{i-1,+}+A_cU_{j-1}^{i,+}+A_dU_j^{i,+}$$
$$=\tau\left(1-\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i-1,+}+S_j^{i-1,+})+\tau\left(1+\tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S_{j-1}^{i,+}+S_j^{i,+})$$
$$\ldots(71)$$

and
$$A'_a U^{i-1,-}_{j-1} + A'_b U^{i-1,-}_{j} + A'_c U^{i,-}_{j-1} + A'_d U^{i,-}_{j}$$
$$= \tau\left(1 - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S^{i-1,-}_{j-1} + S^{i-1,-}_{j}) + \tau\left(1 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right)(S^{i,-}_{j-1} + S^{i,-}_{j}), \quad \ldots(72)$$

where
$$S^{i-1,+}_{j-1} = \sum_{j=1}^{J} \tfrac{1}{2}\omega(P^{++}CU^{+,i-1} + P^{-}CU^{-,i-1})_{j-1} + (1-\omega)B^{i-1,+}_{j}, \quad \ldots(73)$$
$$S^{i-1,+}_{j} = \sum_{j=1}^{J} \tfrac{1}{2}\omega(P^{++}CU^{+,i-1} + P^{+-}CU^{-,i-1})_{j} + (1-\omega)B^{i-1,+}_{j} \quad \ldots(74)$$

Defining
$$\mathbf{A}^{ab} = \begin{bmatrix} A^j_a & A^{j+1}_b & & \\ & A^{j+1}_a & A^{j+2}_b & \\ & \ddots & \ddots & \\ & A^{J-1}_a & A^J_b & \\ & & A^J_a & \end{bmatrix} \quad \ldots(75)$$

and similarly A^{dc} is defined. We can rewrite equations (71) and (72) as
$$[\mathbf{A}^{dc} - \tau^{+}\mathbf{Q}\boldsymbol{\gamma}^{++}]\,\mathbf{U}^{+}_{i} + [\mathbf{A}^{ab} - \tau^{-}\mathbf{Q}\boldsymbol{\gamma}^{++}]\,\mathbf{U}^{+}_{i-1}$$
$$= (1-\omega)\,\tau\mathbf{Q}\mathbf{B}^{+} + \tau^{+}\mathbf{Q}\boldsymbol{\gamma}^{+-}\mathbf{U}^{-}_{i} + \tau^{-}\mathbf{Q}\boldsymbol{\gamma}^{+-}\mathbf{U}^{-}_{i-1} \quad \ldots(76)$$

and
$$[\mathbf{A}'^{fg} - \tau^{+}\mathbf{Q}\boldsymbol{\gamma}^{--}]\,\mathbf{U}^{-}_{i} + [\mathbf{A}'^{ab} - \tau^{-}\mathbf{Q}\boldsymbol{\gamma}^{--}]\,\mathbf{U}^{-}_{i-1}$$
$$= (1-\omega)\,\tau\mathbf{Q}\mathbf{B}^{-} + \tau^{+}\mathbf{Q}\boldsymbol{\gamma}^{-+}\mathbf{U}^{+}_{i} + \tau^{-}\mathbf{Q}\boldsymbol{\gamma}^{-+}\mathbf{U}^{+}_{i-1}, \quad \ldots(77)$$

where
$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & \ddots & \ddots & \\ & 1 & 1 & \\ & & 1 & \end{bmatrix}, \quad \ldots(78)$$
$$\tau^{+} = \tau\left(1 + \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right), \quad \ldots(79)$$
$$\tau^{-} = \tau\left(1 - \tfrac{1}{6}\frac{\Delta A}{\bar{A}}\right), \quad \ldots(80)$$
$$\boldsymbol{\gamma}^{++} = \tfrac{1}{2}\omega\mathbf{P}^{++}\mathbf{C}, \quad \ldots(81)$$

$$\gamma^{+-} = \tfrac{1}{2}\omega \mathbf{P}^{+-}\mathbf{C}. \quad \ldots(82)$$

Similarly γ^{--}, γ^{-+} are defined.
With

$$\bar{\mathbf{A}}_{ab} = \mathbf{Q}^{-1}\mathbf{A}^{ab}, \quad \ldots(83)$$

$$\bar{\mathbf{A}}_{dc} = \mathbf{Q}^{-1}\mathbf{A}^{dc}, \quad \ldots(84)$$

$$\bar{\mathbf{A}}'_{ab} = \mathbf{Q}^{-1}\mathbf{A}'^{ab}, \quad \ldots(85)$$

$$\bar{\mathbf{A}}'_{dc} = \mathbf{Q}^{-1}\mathbf{A}'^{dc}, \quad \ldots(86)$$

equations (76) and (77) become

$$[\bar{\mathbf{A}}_{dc} - \tau^{+}\gamma^{++}]\,\mathbf{U}_i^{+} + [\bar{\mathbf{A}}_{ab} - \tau^{-}\gamma^{++}]\,\mathbf{U}_{i-1}^{+}$$
$$= (1-\omega)\,\tau\mathbf{B}^{+} + \tau^{+}\gamma^{+-}\mathbf{U}_i^{-} + \tau^{-}\gamma^{+-}\mathbf{U}_{i-1}^{-}, \quad \ldots(87)$$

$$[\bar{\mathbf{A}}'_{dc} - \tau^{+}\gamma^{--}]\,\mathbf{U}_i^{-} + [\bar{\mathbf{A}}'_{dc} - \tau^{-}\gamma^{--}]\,\mathbf{U}_{i-1}^{-}$$
$$= (1-\omega)\,\tau b^{-} + \tau^{+}\gamma^{-+}\mathbf{U}_i^{-} + \tau^{-}\gamma^{-+}\mathbf{U}_{i-1}^{+}. \quad \ldots(88)$$

Now the two pairs of transmission and reflection operators are given by

$$\mathbf{t}(i, i-1) = \mathbf{R}^{+-}[\Delta^{+}\mathbf{A} + \mathbf{r}^{+-}\Delta^{-}\mathbf{C}], \quad \ldots(89)$$

$$\mathbf{t}(i-1, i) = \mathbf{R}^{-+}[\Delta^{-}\mathbf{D} + \mathbf{r}^{-+}\Delta^{+}\mathbf{B}], \quad \ldots(90)$$

$$\mathbf{r}(i, i-1) = \mathbf{R}^{-+}[\Delta^{-}\mathbf{C} + \mathbf{r}^{-+}\Delta^{+}\mathbf{A}], \quad \ldots(91)$$

$$\mathbf{r}(i-1, i) = \mathbf{R}^{+-}[\Delta^{+}\mathbf{B} + \mathbf{r}^{+-}\Delta^{-}\mathbf{D}], \quad \ldots(92)$$

where

$$\Delta^{+} = [\bar{\mathbf{A}}_{cd} - \tau^{+}\gamma^{++}]^{-1}, \quad \ldots(93)$$

$$\Delta^{-} = [\bar{\mathbf{A}}_{ab} - \tau^{+}\gamma^{--}]^{-1}, \quad \ldots(94)$$

$$\mathbf{A} = \tau\gamma^{++} + \bar{\mathbf{A}}_{ab}, \quad \ldots(95)$$

$$\mathbf{B} = \tau\gamma^{+-}, \quad \ldots(96)$$

$$\mathbf{C} = \tau\gamma^{-+}, \quad \ldots(97)$$

$$\mathbf{D} = \tau\gamma^{++} - \bar{\mathbf{A}}'_{dc}, \quad \ldots(98)$$

$$\mathbf{r}^{+-} = \tau\Delta^{+}\gamma^{+-}, \quad \ldots(99)$$

$$\mathbf{R}^{+-} = [\mathbf{I} - \mathbf{r}^{+-}\mathbf{r}^{-+}]^{-1}. \quad \ldots(100)$$

The quantities $\mathbf{r}^{-+}$ and $\mathbf{R}^{-+}$ are obtained by interchanging the +ve and −ve signs.

The transmission and reflection operators developed in equations (89)–(92) can be applied to a monochromatic radiation field in both plane parallel and spherical symmetry. These can be employed to calculate the internal radiation field by using the algorithm described by Peraiah (1984).

References

Castor, J. I. (1972) *Ap. J.* **178**, 779.
Mihalas, D. (1978) *Stellar Atmospheres*, Freeman.
Mihalas, D. & Klein, R. I. (1982) *J. Comput. Phys.* **46**, 97.
Mihalas, D., Kunasz, P. B. & Hummer, D. G. (1976) *Ap. J.* **206**, 515.
Munier, A. & Weaver, R. (1986) Preprint : *Computer Phys. Rep.*
Peraiah, A. (1984) *Methods in Radiative Transfer* (ed. : W. Kalkofen) Cambridge Univ. Press.
Peraiah A. (1986, 1987) *Kodaikanal Obs. Bull.* **5**, 141; *Ap. J.* **317**, June 1.
Peraiah, A. & Varghese, B. A. (1985) *Ap. J.* **290**, 411.